ERRATUM TO: BIRATIONALLY RIGID HYPERSURFACES

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ABSTRACT. This note points out a gap in the proof of the main theorem of the article *Birationally rigid hypersurfaces* published in Invent. Math. **192** (2013), 533–566, and provides a new proof of the theorem.

The statements of Lemma 9.1 and Lemma 9.2 in [dF13] are incorrect (see the remarks at the end of this note for further comments on the errors). The two lemmas are used in the proof of Theorem A, which is the main theorem of [dF13]; more precisely, they are used in the proof of Theorem 7.4 from which Theorem A is deduced. The error in Lemma 9.2 was brought to our attention by János Kollár.

While Lemma 9.2 is not essential for the proof and can be circumvented, the gap left by the error in Lemma 9.1 appears to be more substantial because of the key role that Lemma 9.1 plays in the application of Theorem B in the proof of Theorem A.

In this note, we give a new argument to prove Theorem A which does not use Theorem B. This does not fix, however, the proof of Theorem 7.4, which should therefore be considered unproven. We use the same notation and conventions as in [dF13].

Theorem ([dF13], Theorem A)**.** *For $N \geq 4$, every smooth complex hypersurface $X \subset \mathbb{P}^N$ of degree N is birationally superrigid.*

Proof. We assume that $N \geq 7$ and refer to [dFEM03] for the remaining cases $4 \leq N \leq 6$.

Suppose that $\phi \colon X \dashrightarrow X'$ is a birational map, but not an isomorphism, from X to a Mori fiber space X'. The map is defined by a linear system $\mathcal{H}$ whose members are cut out by homogeneous forms of some degree r. Let $D, D' \in \mathcal{H}$ be two general elements, and denote

$$c := \mathrm{can}(X, D).$$

Proposition 7.3 of [dF13] implies that $c < 1/r$. On the other hand, Proposition 8.7 of [dF13] implies that the set of points $Q \in X$ such that $e_Q(D) > r$ is finite. It follows that the pair (X, cD) is terminal in dimension one, and hence there is a closed point $P \in X$ such that $\mathrm{mld}(P; X, cD) = 1$. This implies that $\mathrm{mld}(P; X, cD + P) \leq 0$.

Let $Y \subset X$ be a general hyperplane section through the point P, and let $B := D \cap D' \cap Y$. We remark that Y is a smooth hypersurface of degree N in $\mathbb{P}^{N-1}$ and B is a complete intersection subscheme of Y of codimension two. By inversion of adjunction (e.g., see Theorem 6.1 of [dF13]), we have $\mathrm{mld}(P; Y, cB) \leq 0$. This means that (Y, cB) is not log terminal near P. Notice, though, that (Y, cB) is log terminal in dimension one. In fact, we have the following stronger property.

Lemma 1. *The pair $(Y, 2cB)$ is log terminal in dimension one.*

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Proof. Let $C \subset Y$ be any irreducible curve.

Proposition 8.7 of [dF13] implies that the set of points $Q \in X$ such that $e_Q(D \cap D') > r^2$ has dimension at most one. It follows by Proposition 8.5 of [dF13] that, for a general choice of Y, the set of points $Q \in Y$ such that $e_Q(B) > r^2$ is zero dimensional. Therefore we have $e_Q(B) \le r^2$ for a general point $Q \in C$.

Fix such a point $Q \in C$, and let $S \subset Y$ be a smooth surface cut out by general hyperplanes through Q. By Proposition 8.5 of [dF13], we have $e_Q(B|_S) \le r^2$. Since $B|_S$ is a zero-dimensional complete intersection subscheme of S, the multiplicity $e_Q(B|_S)$ is equal to the Hilbert–Samuel multiplicity of the ideal $\mathcal{I}_{B|_S,Q} \subset \mathcal{O}_{S,Q}$ locally defining $B|_S$ near Q. Then Theorem 0.1 of [dFEM04] implies that the log canonical threshold of $(S, B|_S)$ near Q satisfies the inequality

$$\operatorname{lct}_Q(S, B|_S) \ge \frac{2}{\sqrt{e_Q(B|_S)}}.$$

Since $e_Q(B|_S) \le r^2$ and $c < 1/r$, this implies that $\operatorname{lct}_Q(S, B|_S) > 2c$, and hence $(S, 2cB|_S)$ is log terminal near Q. It follows by inversion of adjunction that $(Y, 2cB)$ is log terminal near Q. As Q was chosen to be a general point of an arbitrary curve C on Y, we conclude that $(Y, 2cB)$ is log terminal in dimension one. □

The lemma implies that the multiplier ideal $\mathcal{J}(Y, 2cB)$ defines a zero-dimensional subscheme $\Sigma \subset Y$. We have $H^1(Y, \mathcal{J}(Y, 2cB) \otimes \mathcal{O}_Y(2)) = 0$ by Nadel's vanishing theorem, since ω_Y is trivial, B is cut out by forms of degree r, and $2cr < 2$. It follows that there is a surjection

$$H^0(Y, \mathcal{O}_Y(2)) \twoheadrightarrow H^0(\Sigma, \mathcal{O}_\Sigma(2)) \cong H^0(\Sigma, \mathcal{O}_\Sigma)$$

(here $\mathcal{O}_\Sigma(2) \cong \mathcal{O}_\Sigma$ because Σ is zero dimensional), and therefore we have

$$h^0(\Sigma, \mathcal{O}_\Sigma) \le h^0(Y, \mathcal{O}_Y(2)) = \binom{N+1}{2}. \tag{1}$$

Lemma 2. *There exists a prime divisor E over X with center P and log discrepancy*

$$a_E(X, cB + P) \le 0$$

such that the center of E in the blow-up of X at P has positive dimension.

Proof. Recall that $\operatorname{mld}(P; Y, cB) \le 0$. We fix a log resolution $f\colon Y' \to Y$ of $(Y, B + P)$, and take a general hyperplane section $Z \subset Y$ through P. Let $Z' \subset Y'$ be the proper transform of Z. By Bertini's theorem, we can ensure that Z' intersects transversally the exceptional locus of f and the induced map $Z' \to Z$ is a log resolution of $(Z, B|_Z + P)$.

We have $\operatorname{mld}(P; Z, cB|_Z) \le 0$ by inversion of adjunction. This means that there is a prime exceptional divisor $F \subset Z'$ with center P in Y and log discrepancy $a_F(Z, cB|_Z) \le 0$. There is a unique prime exceptional divisor $E \subset Y'$ such that F is an irreducible component of $E|_{Z'}$. Note that $E|_{Z'}$ is reduced. Since E is the only prime divisor of Y' that is contained in either supports of the inverse images of B and P and whose restriction to Z' contains F, we have $\operatorname{val}_E(B) = \operatorname{val}_F(B|_Z)$ and $\operatorname{val}_E(P) = \operatorname{val}_F(P)$. It follows by adjunction formula that

$$a_E(Y, cB + P) = a_F(Z, cB|_Z) \le 0.$$

We deduce from the fact that (Y, cB) is log terminal in dimension one that the center of E in Y is equal to P. The fact that $E \cap Z' \ne \emptyset$ (for a general hyperplane section

$Z \subset Y$ through P) implies that the center of E on the blow-up of Y at P is positive dimensional. $\square$

Let E be as in Lemma 2, and let

$$\lambda := \frac{\mathrm{val}_E(P)}{c\,\mathrm{val}_E(B)}.$$

In the next two lemmas, we establish opposite bounds on λ. The proof of the theorem will result by comparing the two bounds.

Lemma 3. $\lambda > \dfrac{1}{N+1}$.

Proof. Let $x, y \in \mathfrak{m}_{Y,P}$ be two general linear combinations of a given regular system of parameters of Y at P. Since the center of E on the blow-up of Y at P is positive dimensional, by taking x, y general we can ensure that $\mathrm{val}_E(f) \le \deg(f)\,\mathrm{val}_E(P)$ for any nonzero polynomial $f(x, y)$.

Let d be any positive integer such that

$$d\,\mathrm{val}_E(P) \le -a_E(Y, 2cB).$$

For every nonzero polynomial $f(x, y)$ of degree $\le d$, we have $\mathrm{val}_E(f) \le -a_E(Y, 2cB)$, and therefore $f \notin \mathcal{J}(Y, 2cB) \cdot \mathcal{O}_{Y,P}$. This means that if $V \subset \mathcal{O}_{Y,P}$ is the $\mathbb{C}$-vector space spanned by the polynomials in x, y of degree $\le d$, then the quotient map $\mathcal{O}_{Y,P} \to \mathcal{O}_{\Sigma,P}$ restricts to a injective map $V \hookrightarrow \mathcal{O}_{\Sigma,P}$, and therefore

$$h^0(\Sigma, \mathcal{O}_\Sigma) \ge \dim_{\mathbb{C}} V = \binom{d+2}{2}.$$

Comparing this inequality with the upperbound on $h^0(\Sigma, \mathcal{O}_\Sigma)$ obtained in (1), we conclude that $N > d$. It follows by our assumption on d that

$$N\,\mathrm{val}_E(P) > -a_E(Y, 2cB).$$

This means that $a_E(Y, 2cB - NP) > 0$. Note, on the other hand, that

$$a_E(Y, 2cB - NP) = a_E(Y, (2 - (N+1)\lambda)cB + P).$$

Since $a_E(Y, cB + P) \le 0$ by Lemma 2, we conclude that $(N+1)\lambda > 1$. $\square$

Lemma 4. $\lambda < \dfrac{\sqrt{N} - 2}{\sqrt{N}(N-5)}$.

Proof. First, we observe that $(N-5)\lambda \le 1$. In fact, since $\mathrm{lct}_P(Y, P) = N - 2$, we have

$$a_E(Y, (N-3)\lambda cB + P) = a_E(Y, (N-2)P) \ge 0,$$

and since $a_E(Y, cB + P) \le 0$ by Lemma 2, we actually get $(N-3)\lambda \le 1$.

Let $S \subset Y$ be a surface cut out by $N - 4$ general hyperplane sections through P. Note that $B|_S$ is a complete intersection zero-dimensional subscheme of S cut out by two forms of degree r. We have

$$a_E(Y, (1 - (N-5)\lambda)cB + (N-4)P) = a_E(Y, cB + P) \le 0.$$

By our initial remark, the pair in the left hand side is effective. We can therefore apply inversion of adjunction, which gives $\mathrm{mld}(P; S, (1 - (N-5)\lambda)cB|_S) \le 0$. This means that

$$\mathrm{lct}_P(S, B|_S) \le (1 - (N-5)\lambda)c.$$

By contrast, by using Theorem 0.1 of [dFEM04], Bezout's theorem, and the inequality $c < 1/r$, we get the chain of inequalities

$$\operatorname{lct}_P(S, B|_S) \geq \frac{2}{\sqrt{e_P(B|_S)}} \geq \frac{2}{r\sqrt{N}} > \frac{2c}{\sqrt{N}}.$$

The lemma follows by comparing the two bounds on $\operatorname{lct}_P(S, B|_S)$. □

To conclude the proof of the theorem, we just observe that the inequalities in Lemmas 3 and 4, combined, imply that $N - 3\sqrt{N} + 1 < 0$, a condition that is never satisfied if $N \geq 7$. □

We close this note with some comments on the errors in Lemmas 9.1 and 9.2 of [dF13].

Remark 1. Lemma 9.1 already fails in the following simple situation. Let $\sigma\colon \mathbb{A}^2 \to \mathbb{A}^1$ be the projection given by $\sigma(x, y) = x$, let $P \in \mathbb{A}^2$ be the origin in the coordinates (x, y), and let $P' = \sigma(P)$. Let $X = (y + x^2 + y^2 = 0) \subset \mathbb{A}^2$, and consider the divisor $E = [P]$ on X. Note that $\mu = \operatorname{val}_E(P) = 1$ and $W = W^1(E)$ is the fiber of $J_\infty X \to X$ over P. Moreover, $\operatorname{val}_E|_{\mathbb{C}(\mathbb{A}^1)} = \operatorname{val}_{E'}$ where $E' = [P']$, and $W' = W^1(E')$ is the fiber of $J_\infty \mathbb{A}^1 \to \mathbb{A}^1$ over P'. In particular,

$$((W')^0)_1 = (W')_1 = T_{P'}\mathbb{A}^1.$$

In the coordinates $(x, y, x', y', x'', y'', \dots)$ of $J_\infty \mathbb{A}^2$, the ideal of W contains the elements $x, y, y', y'' + 2(x')^2$. Fix any integer $m \geq 2$. Since the element $y'' + 2(x')^2$ is in the ideal of W_m, after taking the degeneration to homogeneous ideals as in the proof of Lemma 9.1, the ideal of $(W_m)^0$ contains the elements $x, y, y', (x')^2$. Therefore $((W_m)^0)_1 = \{0\} \subset T_P\mathbb{A}^2$ (set-theoretically), and hence

$$\sigma_1\big(((W_m)^0)_1\big) = \{0\} \in T_{P'}\mathbb{A}^1.$$

This shows that the lemma does not hold in this case.

The error in the proof of Lemma 9.1 is in the last formula. The formula is true before taking closures (namely, we have $\pi_{m,0}^{-1}(P) \cap T_P^{(m)}\mathbb{A}^n = T_P^{(m)}X$), but after taking closures in the projective space we only get an inclusion $\overline{\pi_{m,0}^{-1}(P)} \cap \overline{T_P^{(m)}\mathbb{A}^n} \supset \overline{T_P^{(m)}X}$. In the example discussed above, for instance, the point $(0 : 0 : 0 : 0 : 1)$ in the homogeneous coordinates $(u : x' : y' : x'' : y'')$ belongs to the closure of W_2 (which is the same as $\pi_{2,0}^{-1}(P)$), but not to the closure of $T_P^{(2)}X$.

Remark 2. The error in the proof of Lemma 9.2 is in the wrong assertion that the image under a finite morphism of a Cohen–Macaulay scheme is Cohen–Macaulay. This fails for instance for general projections to $\mathbb{P}^4$ of most projective surfaces in $\mathbb{P}^5$.

References

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